Math 115A, Lecture 2
Linear Algebra

Midterm 1

Instructions: You have 50 minutes to complete the exam. There are five problems, worth a total of fifty points. You may not use any books or notes. Partial credit will be given for progress toward correct proofs.

Write your solutions in the space below the questions. If you need more space use the back of the page. Do not forget to write your name in the space below.

Name: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

Problem 1.

Decide whether each of the following sets V with the operations of addition and scalar multiplication specified is a vector space. Justify your answers.

(a) [5pts.] $V \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$ is the set of 2×2 matrices with determinant zero, and the operations inherited from $\mathrm{Mat}_{2\times 2}(\mathbb{R})$. (Recall that if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix, the determinant is $ad - bc$.)

(b) [5pts.] $V = \{(a_{,}a_2) : a_1, a_2 \in \mathbb{R}\}$ with operations

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= (a_1 + 2b_1, a_2 - 3b_2) \\ c(a_1, a_2) &= (ca_1, c^2 a_2) \end{aligned}$$

Problem 2.

Consider the subset V of polynomials $P_2(\mathbb{R})$ such that, for any $ax^2 + bx + c$ in V, we have $a + b + c = 0$.

(a) [5pts.] Prove that V is a subspace of $P_2(\mathbb{R})$.

(b) [5pts.] Find the dimension of V.

Problem 3.

Consider the set $S = \{(2,3,5),(1,0,-1),(-2,1,7),(1,4,11)\} \subset \mathbb{R}^3$.

(a) [5pts.] Is S linearly independent or linearly dependent? Justify your answer *without doing a computation.*

(b) [5pts.] Find a subset of S that is a basis for $\mathbb{R}^3$.

Problem 4.

Let S_1 and S_2 be subsets of a vector space V.

(a) [5pts.] Prove that $\mathrm{span}(S_1 \cap S_2) \subset \mathrm{span}(S_1) \cap \mathrm{span}(S_2)$.

(b) [5pts.] Give an example in which the sets above are equal and one in which they are unequal.

Problem 5.

Recall that if W_1 and W_2 are subspaces of a vector space V, then

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

If in addition $W_1 \cap W_2 = \emptyset$, then we call this space $W_1 \oplus W_2$. If $W_1 \oplus W_2 = V$, then W_2 is said to be a complement of W_1.

(a) [5pts.] Prove that the xy-plane and the z-axis are complements in $\mathbb{R}^3$.

(b) [5pts.] Let V be an n-dimensional vector space, and W_1 a k-dimensional subspace of V. Prove that W_1 has a complement; that is, prove that there exists W_2 such that $W_1 \oplus W_2 = V$. [Hint: Start with a basis for W_1, and extend to a basis for V. Now you should be able to find a candidate basis for W_2.]